DENSITY-DEPENDENT DISCRETE-TIME S-I-S EPIDEMIC MODELS

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Infectious Disease Models

- R. M. Anderson and R. M. May (1996)
- N. T. J. Bailey (1975)
- W. O. Kermack and A. G. McKendrick (1932)
- R. Ross (1911)
- etc

Questions

What are the consequences of the interactions between density-dependent birth or recruitment processes and disease-induced mortality in seasonal environments?

- (L. Allen [1994, 2000], Hwang and Kuang [2001, 2003], Castillo-Chavez et al [2005], etc).

Demographic Equation (Constant Environment)

$$N(t+1) = f(N(t)) + \gamma_1 S(t) + \gamma_2 I(t) \qquad (1)$$

where at generation t,

$S(t) =$ susceptible population;

$I(t) =$ infected population (assumed infectious);

$N(t) = S(t) + I(t) =$ total population;

$f \in C^1([0,\infty),[0,\infty)$ models birth or recruitment process;

Disease Induced Mortality: $\gamma_1 \geq \gamma_2$

When $\gamma = \gamma_1 = \gamma_2$, then (1) becomes

$$N(t+1) = f(N(t)) + \gamma N(t) \qquad (2)$$

Examples Of Recruitment Functions

1. Constant recruitment function

$$f(N(t)) = \Lambda$$

2. Geometric recruitment function

$$f(N(t)) = \mu N(t)$$

3. Beverton-Holt recruitment function

$$f(N(t)) = \frac{\mu k N(t)}{k + (\mu - 1)N(t)}$$

4. Ricker recruitment function

$$f(N(t)) = N(t)e^{r(1-\frac{N(t)}{k})}$$

Demographic Equation In Seasonal Environments

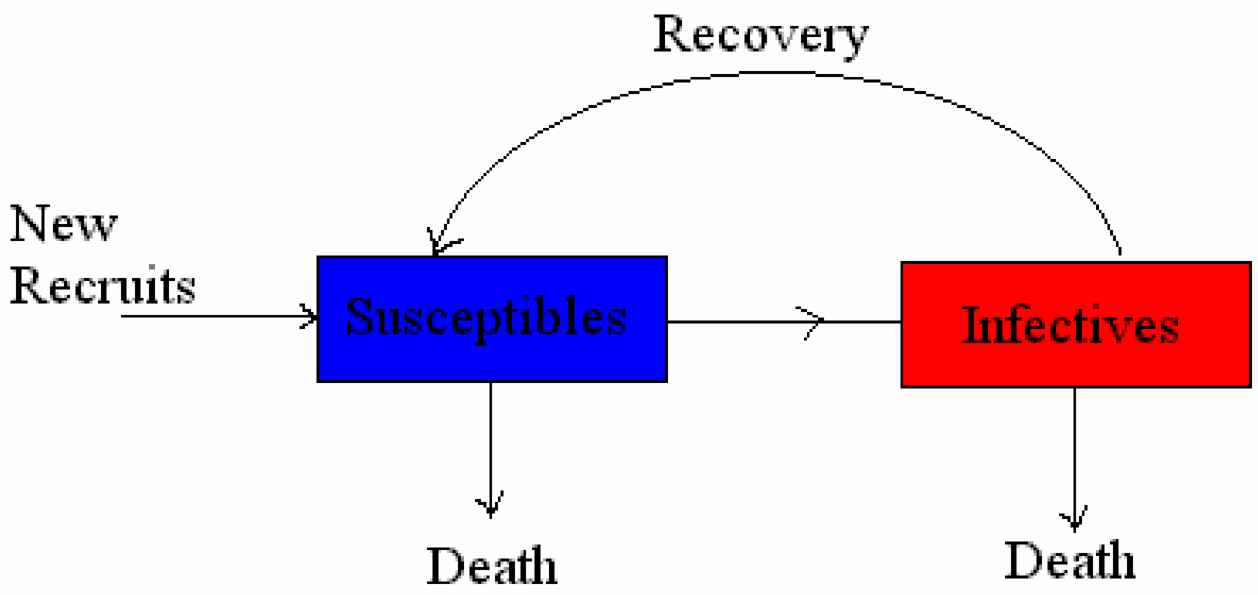
$$N(t+1) = f(t, N(t)) + \gamma_{1t} S(t) + \gamma_{2t} I(t)$$

where

$$f(t+T, N(t)) = f(t, N(t))$$

$$\gamma_{i(t+T)} = \gamma_{it}$$

S-I-S Model


Recovery
New
Recruits
Susceptibles
Infectives
Death
Death

SIS Epidemic Model With Disease-Induced Death

$$\left.\begin{aligned} S(t+1) &= f(N(t)) + \gamma_1 \phi(\alpha \frac{I(t)}{N(t)}) S(t) + \gamma_2 (1-\sigma) I(t) \\ I(t+1) &= \gamma_1 \left(1 - \phi(\alpha \frac{I(t)}{N(t)}) \right) S(t) + \gamma_2 \sigma I(t) \end{aligned}\right\} (3)$$

where $0 < \gamma_2 < \gamma_1 < 1, 0 < \sigma < 1$ and $N(t) > 0$.

The escape function $\phi : [0, \infty) \to [0,1]$ is a monotone convex probability function with $\phi(0) = 1$ and $\phi' \leq 0$.

Model Assumptions

- Disease increases mortality but does not affect fecundity;
- No acquired immunity;
- No latent period (or latent period is very short);
- Transmission is frequency dependent rather than density dependent.

Deterministic SIS Model

- Our model is a deterministic SIS epidemic model and has no "probability" of transmission. The assumption of deterministic dynamics is valid in a large population, where stochasticity is unimportant.
- This assumption places a constraint on the applicability of our model. For example, stochastic transmission (including a Poisson process) in a small population (close to extinction) would not be described by our model.

Disease Extinction or Persistence

Let $R_0 = \frac{-\gamma_1 \alpha \phi'(0)}{1-\gamma_2 \sigma}$.

No disease induced death : Castillo – Chavez and Yakubu [2001]

Theorem (Franke and Yakubu, 2008) :

Let $N(0) \geq I(0) > 0$.

1. If $R_0 < 1$, then $\lim_{t\to\infty} I(t) = 0$. That is, the disease goes extinct.

2. If $R_0 > 1$ and the total population is uniformly persistent, then there exists $\eta > 0$ such that $\underline{\lim}_{t\to\infty} I(t) \geq \eta > 0$. That is, the disease is uniformly persistent.

R_0

- Without disease-induced mortality, it is known that $R_0 > 1$ implies disease persistence.
- With disease-induced mortality, independent of initial population size of healthy individuals, a tiny number of infectious individuals can drive the total population to extinction.

Auxiliary Functions

$$1.\ D_i(N) = f(N) + \gamma_i N$$

The total population of new births and survivors;

$$2.\ \mathrm{F_N}(I) = \gamma_1(1-\phi(\alpha\frac{I}{N}))(N-I) + \gamma_2\sigma I$$

Infective population in the next generation;

$$3.\ \mathrm{G_N}(I) = f(N) + \gamma_1(N-I) + \gamma_2 I$$

Total population in the next generation;

$$4.\ \mathrm{H(N,I)} = (\mathrm{G_N}(I), \mathrm{F_N}(I))$$

Vector of the total and infective populations.

Disease-Free State

If $I(t) = 0$, then the demographic equation

$$N(t+1) = f(N(t)) + \gamma_1 S(t) + \gamma_2 I(t)$$

becomes

$$S(t+1) = f(S(t)) + \gamma_1 S(t).$$

This reduced equation describes the disease-free state dynamics.

Demographic Basic Reproduction Number

$$R_{D_i} = \frac{f'(0)}{1-\gamma_i} \text{ whenever } f(0) = 0.$$

1. Let $f(0) = 0$. If $R_{D_1} > 1$, then the disease-free susceptible population is persistent.

2. Let $f(0) = 0$. If $R_{D_1} < 1$, then $\{(0,0)\}$ is locally asymptotically stable. That is, both the susceptible and infected populations go extinct at low population sizes.

3. R_{D_1} is the disease-free state demographic basic reproduction number.

4. If either $f(0) > 0$ or $f(0) = 0$ and $R_{D_2} > 1$ then the total population is uniformly persistent.

Dramatic Population Extinction

Theorem : Let $R_0 > 1$, $f(0) = 0$ and $f(N) \leq f'(0)N$ for all $N > 0$. Then there is a function $\zeta = \zeta(\gamma_1, \gamma_2, \phi, \alpha, \sigma, F_1) > 1$ such that if $1 < R_{D_1} < \zeta$ then the total population goes extinct under H iterations.

Illustrative Example

Let $f(N) = \dfrac{aN}{1+bN}$ and $\phi(N) = e^{-\frac{\alpha I}{N}}$

where

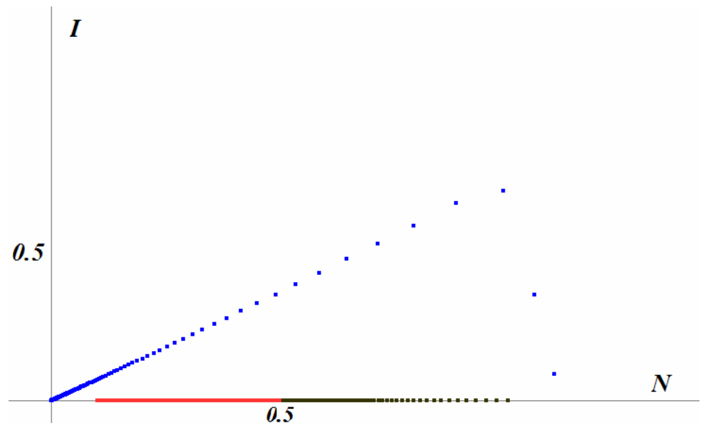
$0.1 < a < 0.2, b = 1, \alpha = 5, \gamma_1 = 0.9,$

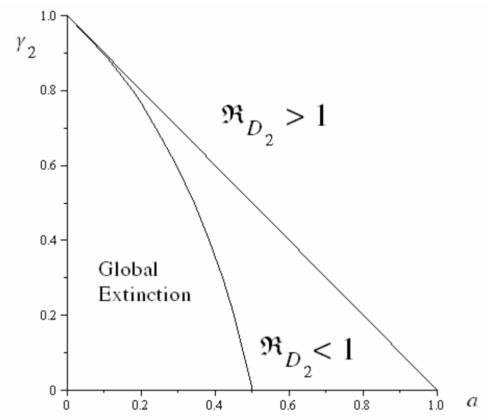
$\gamma_2 = 0.8$ and $\sigma = 0.9.$

$R_{D_1} = \dfrac{a}{1-\gamma_1} > \dfrac{0.1}{1-0.9} = 1$ implies the persistence of the susceptible population in the absence of the disease.

$R_{D_2} = \dfrac{a}{1-\gamma_2} < \dfrac{0.2}{1-0.8} = 1.$

As predicted by the theorem, $0.1 < a < 0.177$ gives the extinction of the total population.





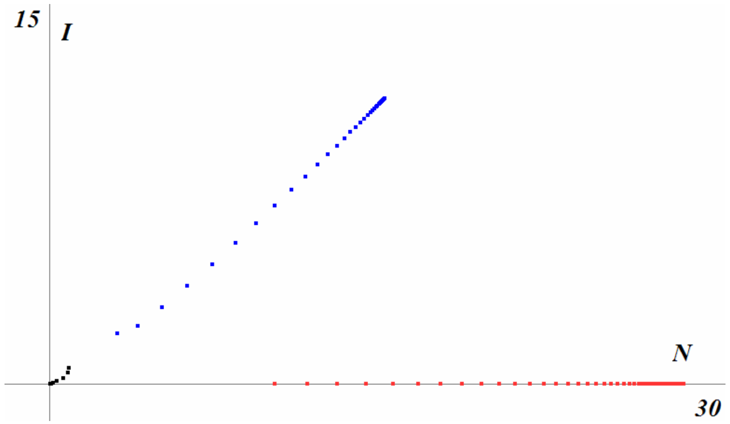
Multiple Attractors

Theorem : Let $\overline{\lim}_{N\to\infty} \dfrac{f(N)+\gamma_1 N}{N} < 1$

and $R_{D_2} > 1$. Then H has multiple fixed points

when G_N "decreases" at low population sizes

while it "increases" at high population values.



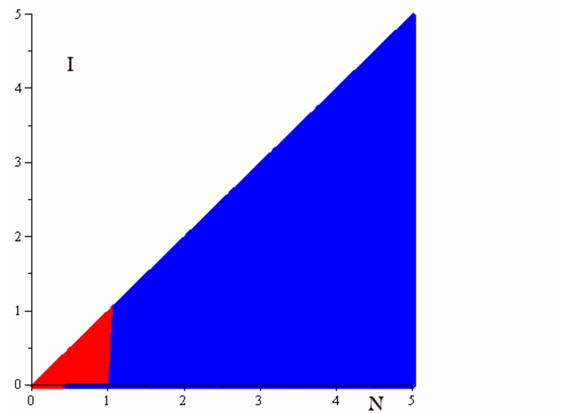
Corollary : Let $\overline{\lim}_{N\to\infty} \dfrac{f(N)+\gamma_1 N}{N} < 1.$

If $R_{D_1} > 1$ and there is $0 < N_0$ with

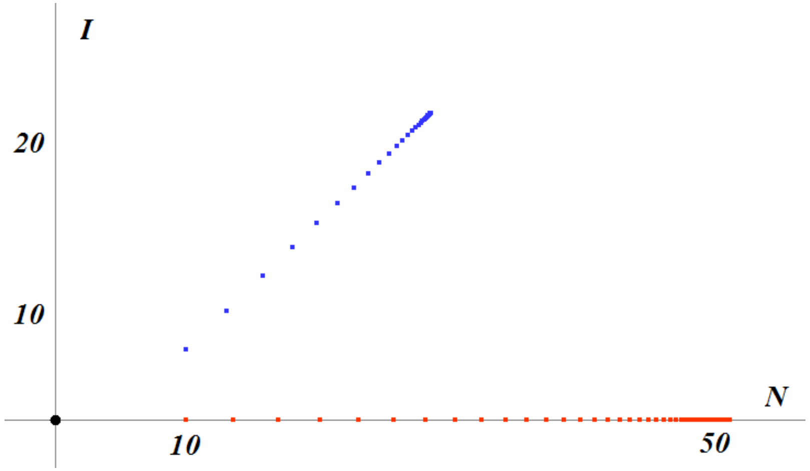
$G_{N_0}(I_1 N_0) > N_0$, then the origin is

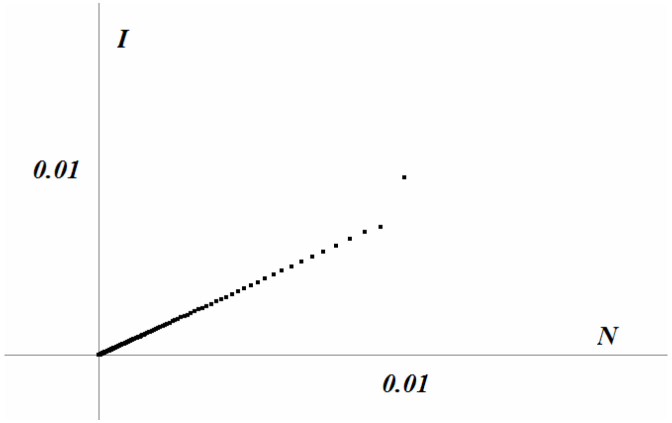
not a global attractor and H has

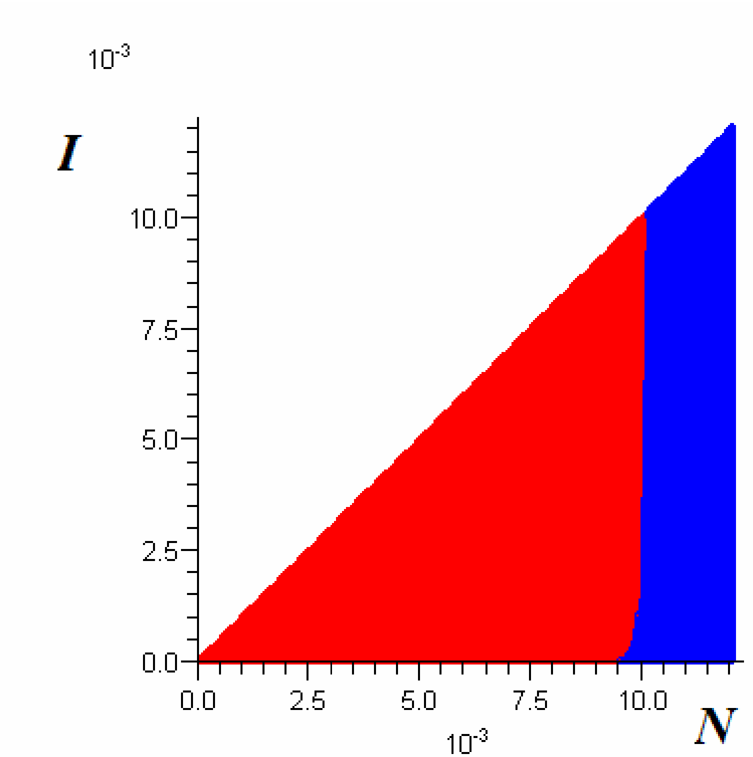
at least two positive fixed points.



Origin an Attractor







Impact of Seasonality

Consider Model (5) with the Beverton-Holt recruitment function

$$f(t, N) = \frac{aN}{1 + b_t N},$$

and

$$\phi\left(\frac{\alpha I}{N}\right) = e^{-\frac{\alpha I}{N}},$$

where

$$0.11 \quad < \quad a < 0.15, \quad b_t = 1.2 + (-1)^t * 0.1, \quad \alpha = 5, \quad \gamma_{1t} = 0.9 + (-1)^t * 0.0$$
$$\gamma_{2t} \quad = \quad 0.8 + (-1)^t * 0.05, \text{ and } \sigma = 0.9.$$

In this example, $\mathcal{R}_{D_i} = \prod_{t=0}^{p-1} (f'(t, 0) + \gamma_{it})$.

$$\mathcal{R}_{D_1} = (a + \gamma_{1,0})(a + \gamma_{1,1}) > (0.11 + 0.95)(.11 + 0.85) > 1.01 > 1$$

implies the persistence of the susceptible population in the absence of the disease (Lemma (8)), where

$$\mathcal{R}_{D_2} = (a + \gamma_{2,0})(a + \gamma_{2,1}) < (0.15 + 0.85)(.15 + 0.75) = 0.9 < 1.$$

With our choice of parameters, the disease-free dynamics are governed by the Beverton-Holt model and the susceptible population persists.

$$a \quad = \quad 2, \quad b_t = 1.3 + (-1)^t * 1.2995, \quad \alpha \in [5, 300], \quad \gamma_{1t} = 0.45 + (-1)^t * 0.03,$$
$$\gamma_{2t} \quad = \quad 0.4 + (-1)^t * 0.02, \text{ and } \sigma = 0.0002.$$

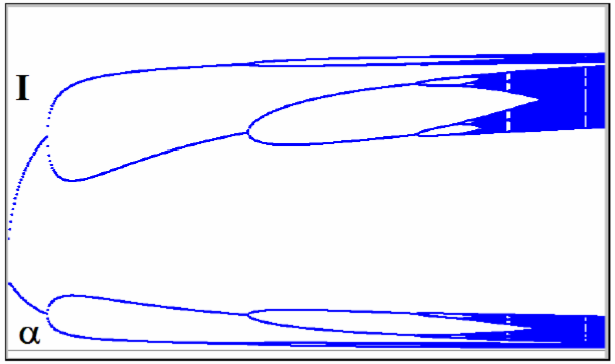


Figure 2: Infective population undergoes period-doubling bifurcation route to chaos as α varies between 5 and 400. On the $x - axis$, $\alpha \in [5, 300]$ and on the $y - axis$, $I \in [0, 400]$.

SIS MODEL IN SEASONAL ENVIRONMENTS

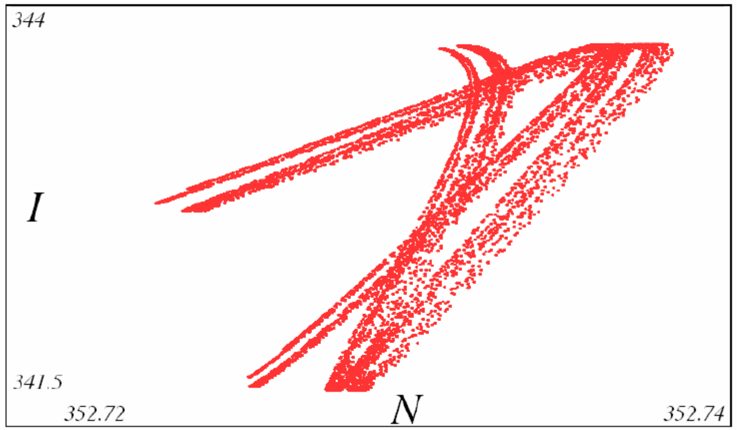


Figure 3: A chaotic attrator in the (I, N) $-space$.

Question

What empirical evidences do we have regarding the potential roles of seasonal fluctuations in cycling (host or pathogen) populations?

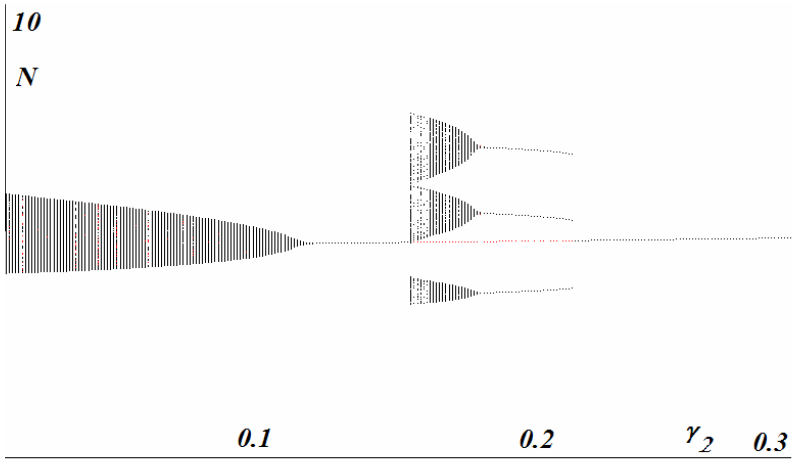
Complex Disease Dynamics

Let $f(N) = N\exp(r - N)$ and $\varphi(\frac{\alpha I}{N}) = e^{-\frac{\alpha I}{N}}$

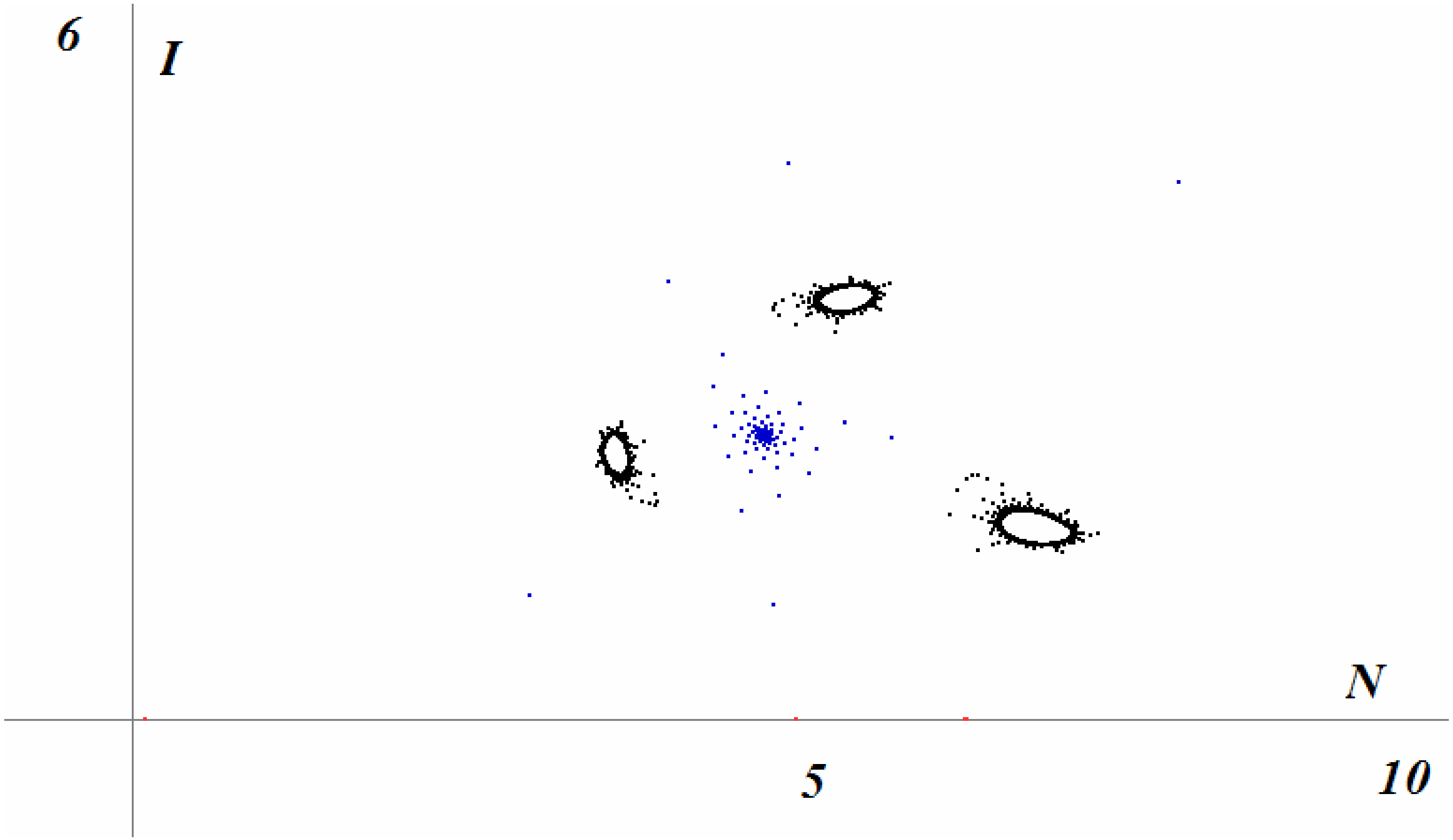
where

$\alpha = 5,\ \gamma_1 = 0.9, \gamma_2 \in (0, 0.9), r = 4$ and $\sigma = 0.9.$

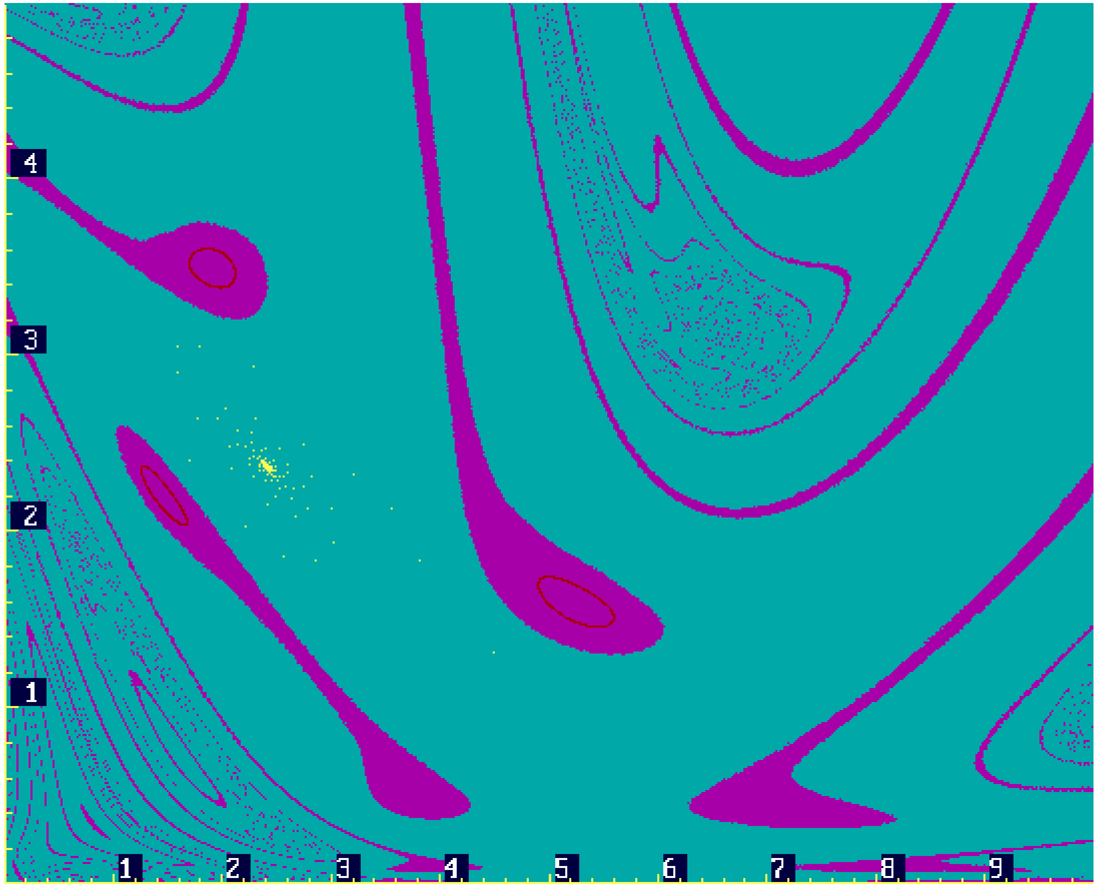
In the absence of the disease, the susceptible population is on a globally attracting positive fixed point at $S_\infty = 6.303.$



Overcompensatory Dynamics



Fractal Basin Boundaries



Geometric Growth

Let $f(N) = \mu N$. In the absence of the disease,

the suceptible (disease - free state) equation

becomes

$$S(t+1) = \mu S(t) + \gamma_1 S(t) = (\mu + \gamma_1) S(t).$$

Hence,

$$S(t) = (\mu + \gamma_1)^t S(0) \text{ and } R_{D_1} = \frac{\mu}{1 - \gamma_1}.$$

SIS Model With Geometric Growth

Let $i = \frac{I}{N}$ and $s = \frac{S}{N}$.

Then $i(t) + s(t) = 1$ and our SIS model becomes

$$i(t+1) = \frac{F_1(i(t))}{\mu + \gamma_1 + (\gamma_2 - \gamma_1)i(t)} \qquad (5)$$

R_0

Under geometric growth,

$$R_0 = \frac{-\gamma_1 \alpha \varphi'(0)}{(1-\gamma_1)(R_{D_1}-1)+1-\gamma_2\sigma}.$$

Theorem: If $R_0 \le 1$, then $\lim_{t\to\infty} i(t) = 0.$
That is, the proportion of the infected eventually decreases to zero.

If $R_0 > 1$, then the proportion of the infected population is uniformly persistent.

Envelopes on Compact Intervals [Cull, 1986]

Let $F:[0,1]\rightarrow[0,1]$ have a unique critical point, i_c, and a unique positive fixed point, i_∞, where $0 < i_c < i_\infty < 1$. Also, let $\{0\}$ be an unstable fixed point of F.

A function $E:[0,1]\rightarrow[0,1]$ envelopes the function F if and only if

$E(i) \geq F(i)$ on $[0, i_\infty]$ and

$E(i) \leq F(i)$ on $[i_\infty, 1]$.

Globally Stable Positive Fixed Point

Theorem (Cull [1986]) : If E envelopes F on $[0,1]$ and $E(E(i)) > i$ for all i in $[i_c, i_\infty)$, then i_∞ is a globally asymptotically stable positive fixed point of F on $(0,1]$.

Theorem [F - Y, 2008] : If $R_0 > 1$, our SIS epidemic model with geomteric growth has a unique positive globally asymptotically stable equlibrium.

Conclusion

- Our model framework allows the population dynamics and disease transmission to be fairly general.
- We highlighted the role of disease-induced mortality, seasonality and the complexity of the interaction between infectives and susceptible in discrete-time models.
- Disease-induced death can force the extinction of a population with $R_0 > 1$, where the population persists without disease-induced death.
- Disease-induced death can generate multiple attractors with complicated basin structures.
- In epidemic models with disease-induced death, the disease-free dynamics do not drive the disease dynamics.
- Seasonal environments can generate complex bifurcations where none existed in constant environments.